The $(QED)_{0+1}$ model
and a possible dynamical solution
of the strong CP problem

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Abstract. The $(QED)_{0+1}$ model describing a quantum mechanical particle on a circle with minimal electromagnetic interaction and with a potential $-M\cos(\varphi - \theta_M)$, so that it mimics the massive Schwinger model, is discussed as a prototype of mechanisms and infrared structures which characterize gauge quantum field theories in positive gauges and QCD in particular. The functional integral representation in terms of the field variables which enter in the Lagrangean displays non–standard features, like a complex functional measure (failure of Nelson positivity), a crucial rôle of the boundary conditions, and the decomposition into θ sectors already in finite volume. In the infinite volume limit, one essentially recovers the standard picture when $M = 0$ ("massless fermions"), but one meets substantial differences for $M \neq 0$: for generic boundary conditions, independently of the lagrangean angle of the topological term, the infinite volume limit selects the sector with $\theta = \theta_M$ and provides a natural "dynamical" solution of the strong CP problem. In comparison with previous approaches, the strategy discussed here allows to exploit the consequences of the θ dependence of the free energy density, with a unique minimum at $\theta = \theta_M$

1. Introduction.

The use of functional integral methods has lead to the discovery [1] [2] [3] of important non–perturbative features of gauge quantum field theories (QFT), in particular the mechanism of θ–vacua, the winding number picture, the $U(1)$ chiral symmetry breaking, the topological terms, the strong CP problem etc. A rigorous analysis of the massive Schwinger model reproducing such features [2] has further backed the standard wisdom on QCD [3].

The problems arise when such picture is confronted with the standard perturbative approach, where it seems difficult to incorporate a non vanishing order parameter $\bar{q}q$ [4] and, in the presence of a fermion mass term, a "natural" strong CP symmetry [5], and the mechanism suggested to solve such problems (non–integer winding numbers, Goldstone dipole, Peccei–Quinn symmetry) are not without difficulties.

The aim of the present note is to revisit the above problems on the basis of a rigorous functional integral analysis of a simple model (which mimics, in $0+1$ dimensions, the massive Schwinger model): as we shall see the CP conserving condition, $\theta = \theta_M$, (θ_M the fermion mass angle) will emerge as a dynamical effect in the thermodynamical limit of the functional integral, generically in the boundary conditions, with no need of fine tuning.

The model (QED_{0+1}) describes a quantum mechanical particle on a circle with minimal electromagnetic interaction and with a potential $-M\cos(\varphi - \theta_M)$, which mimics the fermion mass term in the massive Schwinger model; the coordinate φ is the analog of the scalar field which bosonizes the fermions in $1+1$ dimensions, and that is why it lives on a circle [6]. Without loss of generality, M can be taken non–negative. The model is easily analysed in the Hamiltonian approach,

$$H = \frac{1}{2}(p - eA)^2 + \frac{1}{2}E^2 - M\cos(\varphi - \theta_M) \tag{1}$$

($E = \dot{A}$, $p = \dot{\varphi} + eA$), in terms of the canonical "field" C^* algebra $\mathcal{A}$ generated by $\exp i\varphi$, $\exp i\alpha A$, $\exp i\beta p$, $\exp i\gamma E$, $\alpha, \beta, \gamma \in \mathbb{R}$. For $M > 0$ and small, a unique ground state exists, (its energy has an essential singularity at $e = 0$), it yields a non–regular (i.e. non Schroedinger) representation of $\mathcal{A}$, and a reducible representation of the gauge invariant observable subalgebra $\mathcal{A}_{obs}$ generated by $\exp i\varphi$, $\exp i\beta(p - eA)$, $\exp i\gamma E$, $\beta, \gamma \in \mathbb{R}$. The irreducible representations of $\mathcal{A}_{obs}$ are labelled by an angle θ; the unique ground state of H belongs to the sector $\theta = \theta_M$ and is invariant under the CP symmetry ($\varphi \mapsto -\varphi - 2\theta_M \mod 2\pi$, $A \mapsto -A$).

The so obtained naturality of the CP conserving condition, $\theta = \theta_M$, crucially depends on the strategy (also followed by the standard perturbative approach) of for-

mulating the model in terms of a *field* algebra; the alternative strategy which restricts the attention to the *observable* algebra and its irreducible representations does not allow for a dynamical choice between the various θ sectors, each with a lowest energy state, ψ_θ^0; each value of θ is then allowed and the CP conserving condition becomes accidental; on the other hand, such an approach is intrinsically non–perturbative and the naturality condition (i.e. stability under higher order perturbative corrections) [7] cannot even be posed.

The non–trivial information that among the θ sectors the minimum of the energy is reached for $\theta = \theta_M$ (provided by the first approach) has important consequences also for the functional integral approach, which is done in terms of lagrangean variables. As a matter of fact, the functional integral approach to the model is very instructive since it displays general features and non–standard mathematical properties which are likely to be shared by the functional integral approach to four–dimensional gauge theories, in positive gauges.

The first lesson is that the naive (but popular) euclidean functional integral representation in infinite volume (here infinite time)

$$d\mu\left(\varphi(\tau), A(\tau)\right) = \mathcal{D}x \, \mathcal{D}A \, e^{-\int \left(\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}\dot{A}^2 + ie\dot{\varphi}A - M\cos(\varphi - \theta_M) + i\theta_L\dot{A}\right) d\tau} \equiv$$

$$\equiv d\mu^{free}\left(\varphi(\tau)\right) d\mu^{free}\left(A(\tau)\right) e^{\int \left(-ie\dot{\varphi}A - i\theta_L\dot{A} + M\cos(\varphi - \theta_M)\right) d\tau} \tag{2}$$

(where to be general a topological term, with lagrangean parameter θ_L has been included) is highly misleading, if not wrong. One of the reasons is that it involves infrared singular variables (here the variable A): even in the presence of an ultraviolet cutoff, the ground state defines a non–regular representation of the CCR algebra generated by those variables, and a change of representation takes then place in the infinite volume limit (*infrared renormalization*) [8].

One can show that in finite "volume", $\tau \in [-T, T]$, eq.(2) gives rise to a well–defined measure $d\mu_T$, and in particular there are no ultraviolet problems for the term $\dot{\varphi}A$, contrary to the case of a particle in a magnetic field (here $A = A(\tau)$, rather than $A(x(\tau))$. However, the control of the infinite volume limit $T \to \infty$ exhibits completely new features with respect to ordinary quantum mechanical models, like non–relativistic particles with velocity–independent potentials and scalar field models:

i) $d\mu_T$ is *complex*; Nelson positivity does not hold, so that the continuity of the functional (on the space of continuous functions of the trajectories) which defines $d\mu_T$ (by the Riesz–Markov theorem) is not automatic. Such continuity, which is necessary (and sufficient) for $d\mu_T$ to be a measure, rather than merely a "cylinder measure",

only holds in finite volume. In the infinite volume limit, the correlation functions have a functional integral representation only in terms of a complex cylinder measure, with infinite total variation.

ii) a crucial rôle is played by the Osterwalder–Schrader (OS) positivity condition, also in connection with the problem of *reducibility* on the observable algebra of the functional defined by $d\mu_T$, which gives the θ *angle structure*.

iii) an important rôle is played by the boundary conditions, which may now be complex and are only constrained by OS positivity; they provide the correct way to achieve the reduction into θ sectors, which occurs already *in finite volume*, in terms of functional integrals with winding numbers n and phases $\exp in\theta$ (the popular claim that this is obtained by the gauge invariance condition is misleading, see below and [9]). However, the winding number interpretation loses its meaning in the infinite volume limit.

In agreement with the results of the Hamiltonian approach, the infinite volume limit, $T \to \infty$, of the correlation functions of the observables is strongly affected by the dependence of the "free energy density" on the θ parameter:

a) for $M \neq 0$, *generically in the boundary conditions*, the limit $T \to \infty$ gives the correlation functions on the ground state, which belongs to the sector with $\theta = \theta_M$. This gives a *dynamical solution of the strong CP problem*. The results of 't Hooft analysis [1], [10] can be obtained only by a "fine tuned" choice of the boundary conditions, i.e. by taking an ergodic mean on the boundary variable A with weight $\exp i\theta A$; such a choice ("non–local in A") is unnatural from a perturbative point of view and it is *not* required by gauge invariance (see also [9]).

b) for $M = 0$ one recovers the standard picture [1–3]: all the lowest energy states in the various θ sectors have the same energy (vacuum degeneracy); chiral symmetry is *unbroken* in the representation of the field algebra, but it is *broken* in each θ sector, i.e. in each irreducible representation of $\mathcal{A}_{obs}$; all such representations are mathematically inequivalent but physically equivalent, since they are related by automorphisms of $\mathcal{A}_{obs}$ which commute with the dynamics. Thus, the limits $M \to 0$ and $T \to \infty$ *do not commute*, even if the mass perturbation is well–defined and small in each θ sector; the reduction into θ sectors only takes place in the $M = 0$ case, and its extrapolation to $M \neq 0$ (as implied in the standard picture [1]) is not correct.

In conclusion, for $M = 0$ the picture displayed by the model coincides with the standard wisdom [1–3] and the rigorous analysis of [2] and [11], whereas for $M \neq 0$ it sheads light on the substantial differences which characterize the case of massive fermions. In particular, the model suggests that a "dynamical solution" of the strong

CP problem takes place in the infinite volume limit, in agreement with the arguments presented in for the massive Schwinger model and the QCD case [13].

2. Hamiltonian approach

The model is defined by the "field" algebra $\mathcal{A}$, which can be taken as the C^* algebra generated by $\exp i\varphi$, $\exp i\alpha A$, $\exp i\beta p$, $\exp i\gamma E$, $\alpha, \beta, \gamma \in \mathbb{R}$. As it is typical of field algebra arising from fermion bosonization [6], $\mathcal{A}$ has a non–trivial centre, $\mathcal{Z}_F$, generated by $\exp 2\pi ip$ and therefore, in each irreducible representation of $\mathcal{A}$, $\exp 2\pi ip$ is a complex number, $\exp 2\pi i\theta_F$, $\theta_F \in [0, 1)$. θ_F is a physically unobservable parameter (it plays the rôle of the second angle in the Schwinger model [11]), since different values of θ_F are related by the *gauge* automorphisms of $\mathcal{A}$:

$$\varphi \mapsto \varphi \ , \quad p \mapsto p + \lambda \ , \quad A \mapsto A + \lambda/e \ , \quad E \mapsto E \tag{3}$$

The gauge invariant observable subalgebra $\mathcal{A}_{obs}$ has a non–trivial centre $\mathcal{Z}$ generated by $\exp iq \equiv \exp i(\varphi - E/e)$, and therefore each irreducible representation of $\mathcal{A}_{obs}$ is labelled by the angle θ (θ *sector*), defined by the value $\exp i\theta$ taken by $\exp iq$.

The Hamiltonian H takes a simple form in terms of the new canonical variables $Q \equiv E/e$, $P \equiv p - eA$, $q \equiv \varphi - E/e$, p:

$$H = \frac{1}{2}(P^2 + e^2 Q^2) - M \cos(Q + q - \theta_M) \tag{4}$$

In each θ sector, q can be replaced by θ and the corresponding Hamiltonian, which depends only on $\theta - \theta_M$, will be denoted by H_θ.

The irreducible regular (i.e. Schroedinger) representations of $\mathcal{A}$ are defined in $L^2([0, 2\pi) \times \mathbb{R}, d\varphi \, dA)$, where p acts as $-i\partial/\partial\varphi$ with boundary conditions $\psi(2\pi, A) = \psi(0, A) \exp 2\pi i\theta_F$. Since H is invariant under the gauge transformations (3), its spectrum is independent of θ_F.

For $M = 0$, in $L^2([0, 2\pi) \times \mathbb{R}, d\varphi \, dA)$ the Hamiltonian (1) has only discrete eigenvalues E_k, $k \in \mathbb{N}$, with infinite multiplicity, and the eigenvectors can be labelled by the eigenvalues $n \in \mathbb{Z}$ of p. The lowest energy eigenvectors ψ_n^0, corresponding to $k = 0$, are the strict analogues of the n–vacua in the standard picture of the massless Schwinger model [1–3][11]; they are invariant under *chiral transformations*, defined by

$$\varphi \mapsto \varphi + \mu \bmod 2\pi, \quad p \mapsto p \ , \quad A \mapsto A \ , \quad E \mapsto E \tag{5}$$

which leave H invariant. The vectors ψ_n^0 define reducible representations of $\mathcal{A}_{obs}$, with an integral decomposition over the θ angle

$$\psi_n^0 = \int_0^{2\pi} e^{in\theta} \, \psi_\theta^0 \, d\theta$$

The θ–vacua ψ_θ^0 do not belong to $L^2([0, 2\pi) \times \mathbb{R}, \, d\varphi \, dA)$ (they define non–regular representations of $\mathcal{A}$, i.e. not continuous in β, γ, δ, see below). In each θ sector *chiral symmetry is spontaneously broken.*[1–3][11].

The situation is more intriguing for $M \neq 0$, since H has a purely continuous spectrum in $L^2([0, 2\pi) \times \mathbb{R}, \, d\varphi \, dA)$, so that there is no ground state.

Theorem 1. *Let $\theta_F \in [0, 2\pi)$ be fixed.*
i) There is a unique irreducible representation π_0 of the field algebra $\mathcal{A}$ such that the Hamiltonian H, for $M \neq 0$, is well defined and has a ground state. Such a representation is the only one in which the spectrum of $\exp iq$ is a pure point spectrum.
ii) The Hilbert space $\mathcal{H}$ of π_0 is given by the Gelfand–Naimark–Segal (GNS) construction on the state

$$\Omega_{\theta, \theta_F}(e^{inq} e^{i\beta p} e^{i\gamma Q + \delta P}) = \begin{cases} e^{i(n\theta + m\theta_F)} e^{-(e\gamma^2 + \delta^2/e)/4} & \text{if } \beta/2\pi = m \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

iii) $\mathcal{H}$ can be decomposed as a direct sum of sectors

$$\mathcal{H} = \sum_{\theta \in [0, 2\pi)} \oplus \, \mathcal{H}_\theta \quad (7)$$

The observable algebra $\mathcal{A}_{obs}$ acts irreducibly in each $\mathcal{H}_\theta$. The Hamiltonian leaves each $\mathcal{H}_\theta$ invariant and has in each $\mathcal{H}_\theta$ a pure point spectrum, with no degeneracy. The lowest energy state in $\mathcal{H}$ is unique for $M > 0$ and small, and it belongs to $\mathcal{H}_{\theta = \theta_M}$

Proof. The Fock property of π_0 on the subalgebra $\mathcal{A}_{QP}$ generated by $\exp i(\gamma Q + \delta P)$ is required by the existence of $P^2 + e^2 Q^2$, since $M \cos(Q + q - \theta_M)$ is always a bounded perturbation. Furthermore, if we decompose $\mathcal{H}$ over the spectrum $\exp i\theta$ of $\exp iq$, it follows that the infimum of the spectrum of H_θ, $E^0(\theta)$, has only one minimum, at $\theta = \theta_M$, and therefore the existence of a ground state implies the existence of the discrete component $\mathcal{H}_{\theta_M}$, the irreducibility of the representation then implies the discreteness of the spectrum of $\exp iq$, i.e. equation (7). Any vector $\psi_\theta \in \mathcal{H}_\theta$ defines, by the Fock property of π_0, a Fock state on $\mathcal{A}_{QP}$, and therefore, by applying the algebra $\mathcal{A}_{QP}$ and taking strong limits one can construct a vector ψ_θ^0 which is a

Fock no–particle state for $\mathcal{A}_{QP}$. Now, eq.(6) follows because ψ_θ^0 is an eigenvector of $\exp inq$ and $\exp 2\pi imp$ with eigenvelues $\exp in\theta$, $\exp im\theta_F$, respectively; $e^{i\beta p}$, $\beta/2\pi \notin \mathbb{Z}$, changes the eigenvalue of $\exp iq$, and therefore $e^{i\beta p}\psi_\theta^0$ is orthogonal to all vectors of the form $B\psi_\theta^0$, $B \in \mathcal{A}_{QP}$.

In each sector $\mathcal{H}_\theta$, the state with lowest energy is unique by a Perron-Frobenius argument, since the kernel of $\exp -\tau H_\theta$ in the variable Q is strictly positive. For M small (and fixed charge $e > 0$), the corresponding eigenvalues $E^0(\theta)$ are given by perturbative expansion in M:

$$E^0(\theta) = e/2 - \exp(-\frac{1}{4e})\, M \cos(\theta - \theta_M) + O(M^2) \tag{8}$$

so that, for $M > 0$ the absolute minimum is attained for $\theta = \theta_M$. q.e.d.

$\Omega_{\theta\theta_F}$ admits a unique extension to the Weyl algebra in two degrees of freedom, $\mathcal{A}_{QP} \times \mathcal{A}_{qp}$, which is a Fock state on $\mathcal{A}_{QP}$ and a Zak state [12] on $\mathcal{A}_{qp}$.

The chiral transformations are implementable in π_0 (with p as generator), but spontaneously broken in each irreducible representation $\mathcal{H}_\theta$ of $\mathcal{A}_{obs}$. In each irreducible representation of the field algebra $\mathcal{A}$ (θ_F fixed) the gauge transformations are spontaneously broken, leaving unbroken the discrete subgroup given by $p, A \mapsto p + n, A + n/e$, $n \in \mathbb{Z}$, which is implemented by $\exp inq$. In the analogy with the Schwinger model such unbroken group corresponds to the "large gauge transformations" [1–3],[11]. The algebra $\mathcal{A}$ can be seen as generated by $\mathcal{A}_{obs}$ and the "charged fields" $\exp i\alpha p$, $\alpha \in \mathbb{R}$, which play the rôle of $\exp i\alpha Q_5$ in the Schwinger model. Clearly, for mass $M \neq 0$ there are no n–vacua.

In view of the construction of a functional integral representation of the model, some remarks are necessary on the irreducible (regular) representations of $\mathcal{A}$ in $L^2([0, 2\pi) \times \mathbb{R}, d\varphi\, dA)$; it is easy to see that they are all unitarily equivalent to those in which p acts as $-i\partial/\partial\varphi + \theta_F$, with periodic boundary conditions; this form is the most convenient one for the control of the functional integral, since θ_F does not appear in domain problems, and the kernel of $\exp -tH$, is a periodic function of the angle φ. For each fixed θ_F, the representation of $\mathcal{A}$ in $L^2([0, 2\pi) \times \mathbb{R}, d\varphi\, dA)$ is also unitarily equivalent to that in $L^2(\mathbb{R} \times [0, 2\pi), dQ\, d\theta)$, with $\exp i\beta p$ acting as

$$e^{i\beta p}\phi(Q, \theta) = \phi(Q, \theta + \beta \bmod 2\pi)\, e^{i\beta\theta_F} \tag{9}$$

The unitary equivalence is given by

$$\psi(\varphi, A) = 1/\sqrt{2\pi} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} e^{i(A-\theta_F)(\varphi-\theta-2\pi n)}\, \phi(\varphi-\theta-2\pi n, \theta)\, d\theta \equiv \int_0^{2\pi} \psi_\theta(\varphi, A)\, d\theta$$

$$\tag{10}$$

where ϕ is the wave function in the Q, q representation. Eq.(10) corresponds to the integral decomposition

$$L^2([0, 2\pi) \times \mathbf{R}, \, d\varphi \, dA) = \int_0^{2\pi} d\theta \, \mathcal{H}_\theta \qquad (11)$$

over the spectrum of q. In the following for simplicity we will put $e = 1$.

3. Path integral formulation. Boundary conditions, θ sectors, winding numbers.

One of the main interests of the model is that it can be used as a laboratory for ideas and extrapolations to the functional integral approach to gauge QFT. In this perspective one is confronted essentially with two possible strategies. The first one exploits the possibility of writing the Hamiltonian in terms of gauge invariant fields; one then restricts the attention to the algebra of observables $\mathcal{A}_{obs}$, trivializes its centre (by fixing $\exp iq = \exp i\theta$), writes a functional integral representation for the kernel of $\exp -tH$, and performs the infinite volume limit of the euclidean correlation functions of a maximal abelian subalgebra of $\mathcal{A}_{obs}$; this is equivalent to the use of a functional measure defined on trajectories taking values in the spectrum of $\mathcal{A}_{obs}$ (see also [8]). In this way one constructs all the representations defined by the θ–vacua, for all θ.

This is essentially the strategy followed by [2], where the free paremeter θ which appears in the fermion bosonization in 1+1 dimensions, and enters in the construction of the electric field from the fermion currents, plays the rôle of the variable which describes the spectrum of the centre of $\mathcal{A}_{obs}$.

Similar results are obtained in the functional integral approach to QCD [1–3], where the θ vacua are obtained by summing over the topological number ν with weight $\exp i\nu\theta$. However, in contrast with the case of classical trajectories, the topological classification has no meaning in infinite volume, and therefore a volume "cut–off" is necessary, and boundary conditions must be specified (this is also the rôle of the formulation on the sphere, with regularity at the point at infinity playing the rôle of (special) boundary conditions). In this approach, the choice of the θ sector, and its relation with the sum over topological numbers, relies on a particular choice of boundary conditions [13], and it is really done *before* the infinite volume limit; θ plays therefore the rôle of an external, and free, kinematical constraint, which is independent of any dynamical (energy density) effect, also in the infinite volume limit. This strategy can be realized in the present model, but crucially requires a *special* choice of boundary conditions (see below).

In our opinion it is of interest to explore a strategy which does not rely on low–dimensional pecularities, approaches the construction of the model by functional integrating in finite volume over the field variables which enter the Lagrangean with *generic* boundary conditions, and then takes the infinite volume limit. This alternative gives explicit information on the relevance of boundary conditions and of free energy density effects, which are expected to play an important rôle also in the (standard) four–dimensional functional integral [14] over the gauge potentials $A_\mu(x)$ and the Fermion field $\Psi(x)$, $x \in \mathbb{R}^4$.

Theorem 2.

i) *For any finite interval $[-T, T]$, the formula*

$$d\tilde{\mu}_{x_-,A_-,x_+,A_+,T}(x(\tau), A(\tau)) =$$

$$= d\mu^0_{x_-,x_+,T}(x(\tau)) \, d\mu^0_{A_-,A_+,T}(A(\tau)) \, e^{-i\int_{-T}^{T} \dot{x} A \, d\tau} \, e^{M\int_{-T}^{T} \cos(x - \theta_M) \, d\tau} \tag{12}$$

with $d\mu^0_{\xi_-,\xi_+,T}(\xi(\tau))$ the conditional Wiener measure on paths starting at ξ_- at $\tau = -T$ and ending at ξ_+ at $\tau = T$, defines a complex measure, with finite total variation, absolutely continuous with respect to the free measure $d\mu^0(x(\tau)) \, d\mu^0(A(\tau))$, and therefore supported on trajectories $x(\tau) \in \mathbb{R}$, $A(\tau) \in \mathbb{R}$, which are Hölder continuous of all orders $\alpha < 1/2$. $d\tilde{\mu}$ defines a complex measure $d\mu$ on trajectories $\varphi(\tau) \in S^1 \equiv [0, 2\pi)$, $A(\tau) \in \mathbb{R}$ by the equation

$$d\mu_{\varphi_-,A_-,\varphi_+,A_+,T}(\varphi(\tau), A(\tau)) = \sum_{n=-\infty}^{\infty} d\tilde{\mu}_{\varphi_-,A_-,\varphi_+-2\pi n,A_+,T}(x(\tau), A(\tau)) \tag{13}$$

with $\varphi(\tau) = x(\tau) \mod 2\pi$.

ii) *The measure*

$$d\mu^{\theta_F}_{\varphi_-,A_-,\varphi_+,A_+,T}(\varphi(\tau), A(\tau)) \equiv$$

$$\sum_{n=-\infty}^{\infty} e^{i\theta_F(\varphi_+-2\pi n-\varphi_-)} \, d\tilde{\mu}_{\varphi_-,A_-,\varphi_+-2\pi n,A_+,T}(x(\tau), A(\tau)) \tag{14}$$

coincides with the measure defined, with boundary conditions $\varphi_-, A_-, \varphi_+, A_+$, by the kernel $K^{\theta_F}_\tau(\varphi, A, \varphi', A')$ of $\exp -\tau H$ in $L^2([0, 2\pi) \times \mathbb{R}, d\varphi \, dA)$, where H is defined by eq.(1) with $p = -i\partial/\partial\varphi + \theta_F$ and periodic boundary conditions in φ:

$$K^{\theta_F}_\tau(\varphi, A, \varphi', A') = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\varphi(A - \theta_F)} \, e^{-i(\varphi' - 2\pi n)(A' - \theta_F)}$$

$$\int_{-\infty}^{\infty} e^{-iq(A-A')} \, R_\tau(\varphi - q, \varphi' - q - 2\pi n, q) \, dq \qquad (15)$$

where for any fixed $q \in \mathbb{R}$, $R(Q, Q', q)$ is the (positive) kernel corresponding to the Hamiltonian H_q (eq.(4)).

iii) *Under gauge transformations one has*

$$d\mu_{\varphi_-, A_-, \varphi_+, A_+, T} \left(\varphi(\tau), A(\tau) - \lambda \right) =$$

$$= e^{i[\lambda](\varphi_+ - \varphi_-)} \, d\mu_{\varphi_-, A_- + \lambda, \varphi_+, A_+ + \lambda, T}^{(\lambda \bmod 1)} \left(\varphi(\tau), A(\tau) \right) \qquad (16)$$

$$K_\tau^{\theta_F = 0}(\varphi, A - \lambda, \varphi', A' - \lambda) = e^{i[\lambda](\varphi_+ - \varphi_-)} \, K_\tau^{\theta_F = \lambda \bmod 1}(\varphi, A, \varphi', A') \qquad (17)$$

with $[\lambda]$ the integer part of λ, i.e. $\lambda = [\lambda] + (\lambda \bmod 1)$.

iv) $K_\tau^{\theta_F}$ *defines a bounded (hermitean) semigroup in* $L^2([0, 2\pi] \times \mathbb{R}, \, d\varphi \, dA)$; *it is irreducible in the sense that (for any fixed $\tau > 0$) it does not leave stable any non–trivial subspace of the form* $L^2(B, d\varphi \, dA)$, $B \subset [0, 2\pi) \times \mathbb{R}$.

Proof. We briefly sketch the proof, for more details see [9]. i) One has to control the convergence of the discretizations of the integral $\exp -i \int_{-T}^{T} \dot{x} A \, d\tau$ on trajectories in the support of $d\mu^0(x(\tau)) \times d\mu^0(A(\tau))$, to a result which is independent of the discretization and defines a measurable bounded function of the trajectories [9] (it is sufficient to consider the case $x_- = A_- = 0$). Since also $\exp M \int_{-T}^{T} \cos(x - \theta_M) \, d\tau$ is a measurable bounded function, so is their product. The absolute continuity of $d\tilde{\mu}$ with respect to the Wiener measure and the boundedness of its variation then follow. ii) The kernel of $\exp -\tau H$ in $L^2(\mathbb{R} \times [0, 2\pi], \, dQ \, dq)$ is $R_\tau(Q, Q', q) \, \delta(q' - q)$, with R_τ the kernel of $\exp -\tau H_q$ in $L^2(\mathbb{R})$, H_q being given by eq.(4). $K_\tau^{\theta_F}$ is easily obtained from this kernel by using the unitary transformation in eq.(10). It remains to prove that, for all $\tau > 0$,

$$K_{2\tau}^{\theta_F}(\varphi, A, \varphi', A') = \int d\mu_{\varphi, A, \varphi', A', \tau}^{\theta_F} \left(\varphi(\tau'), A(\tau') \right) \qquad (18)$$

Now, in the r.h.s. of eq.(18), the integral in $d\mu^0(A(\tau))$ is gaussian and the result is

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i\varphi(A - \theta_F)} \, e^{-i(\varphi' - 2\pi n)(A' - \theta_F)} \int_{-\infty}^{\infty} dq \, e^{-iq(A - A')}$$

$$\left[\int d\mu_{\varphi - q, \varphi' - q - 2\pi n, \tau}^{0} \left(x(t) \right) e^{-\frac{1}{2} \int_{-\tau}^{\tau} x^2(t) \, dt} \, e^{M \int_{-\tau}^{\tau} \cos(x(t) - q) \, dt} \right] \qquad (19)$$

The term in square brackets in eq.(19) is exactly the Feynman–Kac representation of the kernel $R_{2\tau}(\varphi - q, \varphi' - q' - 2\pi n, q)$.

iii) Eqs. (16) and (17) are proved by explicit calculation.

iv) is equivalent to the property that $K_\tau^{\theta_F}(\varphi, A, \varphi', A')$ is different from zero almost everywhere. In fact, the integral in the r.h.s. of eq.(15) defines an entire function of A, since it is the Fourier transform in q of R_τ, which decreases faster than any exponential. The sum over n does not destroy analyticity since it converges uniformily in bounded complex domains, by the above decrease properties of R_τ; it follows that the zeros of $K_\tau^{\theta_F}(\varphi, A, \varphi', A')$ are isolated in A, and therefore a set of zero measure with respect to $d\varphi\, dA$.

Remark. Alternatively, given $K_\tau^{\theta_F}(\varphi, A, \varphi', A')$, eq.(15), one could construct the complex measure $d\mu^{\theta_F}(\varphi(\tau), A(\tau))$ starting from the cylinder measure defined by the above kernel (with given boundary values $\varphi_-, A_-, \varphi_+, A_+$); the existence of a measure (complex, of bounded total variation) then follows for $M = 0$ from the estimate

$$\int |K_\tau^{\theta_F}(\varphi, A, \varphi', A')|\, d\varphi'\, dA' \leq e^{\tau/2}$$

which is proved by explicit computation; for $M \neq 0$ the result follows from Trotter's product formula, which also controls the measurability of the mass term with respect to the $M = 0$ measure.

We now discuss the rôle of the boundary conditions and the winding number interpretation of the functional integral on observable variables.

Most of the standard wisdom [1-3] on the functional integral approach to gauge field theory makes use of geometrical ideas which are in general formulated in infinite volume (in analogy with the classical case). As we will also see below, the possibility of incorporating such geometrical structures in a functional integral theoretical setting is problematic in infinite volume, since the relevant measures are not supported on trajectories with definite (classical) behaviour at infinity. In our opinion, it is therefore essential to restrict the use of such structures to finite volume functional integrals, and to discuss i) the rôle of the boundary conditions and ii) the stability of topological structures in the thermodynamical limit.

We recall that, in the $(QED)_{1+1}$ model, the usual analysis, which is actually done in the case of zero fermion mass and then extrapolated to the massive case, relies on 1) the use of (locally regular) euclidean field configurations which are a pure gauge in a neighbourhood of euclidean infinity and regular there, so that they are classified by

a topological number $\nu \in \mathbb{Z}$; at sufficiently large positive and negative times τ_+, τ_-, they are classified by *winding numbers* n_+, n_- ($\nu \equiv n_+ - n_-$);

2) the (formal) existence of (time independent) unitary operators $\mathcal{T}$ which implement the shift $n \to n+1$ and commute with the observables (*large gauge transformations*).

On the basis of 1) and 2) the Hilbert space is decomposed in sectors $\mathcal{H}_\theta$, stable under $\mathcal{A}_{obs}$, which diagonalize $\mathcal{T}$. In terms of functional integrals, the θ sector are formally obtained by adding to the Lagrangean the topological term $\theta\,\varepsilon_{\mu\nu}F^{\mu\nu}$.

The same analysis could be applied to $(QED)_{0+1}$ with the volume replaced by the euclidean time, the regularity at infinity amounting to $A(\pm\tau) \to A(\pm\infty)$, and $\exp 2\pi i A(\infty) = \exp 2\pi i A(-\infty)$, i.e. $A(\infty) - A(-\infty) = \nu$; ν is then the topological invariant of the trajectories $\exp 2\pi i A(\tau) : \dot{\mathbb{R}} \to S^1$, $\dot{\mathbb{R}}$ being the compactified euclidean time. Without loss of generality one can fix $\exp 2\pi i A(\infty) = 1$, so that for large positive and negative times $A(\pm T) \simeq n_\pm$, which play the rôle of the winding numbers. The topological term is $i\theta \int (dA/d\tau)\,d\tau$.

Now, one may show [9] that the infinite volume measure is a complex cylinder measure with infinite total variation, and therefore the set S_{reg} of configurations which are regular at infinity is not even measurable, since the cylinders are defined by the values of the variables lying in finite time intervals. On the other hand, given any extension of the family of measurable sets which includes S_{reg}, the integral of any product of field variables vanishes when restricted to S_{reg} (as a consequence of time translation invariance and cluster property), whenever the measure of S_{reg} can be expressed in terms of countable additivity on cylinder sets from which S_{reg} can be constructed:

$$S_{reg} = \cap_n \cup_{T_1} \cap_{T_2} S_{reg}(T_1, T_2, 1/n)$$

$$S_{reg}(T_1, T_2, \varepsilon) \equiv \{(\varphi(\tau), A(\tau)) : |A(\tau_1) - A(\tau_2)| < \varepsilon \ \forall \tau_1, \tau_2 \in [T_1, T_2]\}$$

$$\mu(S_{reg}) = \lim_{n\to\infty} \lim_{T_1\to\infty} \lim_{T_2\to\infty} \mu(S_{reg}(T_1, T_2, 1/n))$$

We shall actually see that the whole standard picture (n vacua, θ vacua, functional integral decomposition etc.) is problematic in infinite volume, especially in the presence of a fermion mass term. For these reasons, in agreement with the well–estabilished wisdom from Statistical Mechanics, we will first discuss the crucial rôle of the boundary conditions for the functional integral in finite volume, and then perform the infinite volume ("thermodynamical") limit.

The first important feature is that the finite volume euclidean correlation functions, which satisfy the Osterwalder-Schrader (OS) positivity [15], give rise [9] to a

(finite volume) quantum mechanical Hilbert Space $\mathcal{H}_{\mathcal{OS}}$, which carries an irreducible representation of the field algebra $\mathcal{A}$ but in general a reducible representation of the observable algebra $\mathcal{A}_{obs}$. The reducibility of $\mathcal{A}_{obs}$ can be traced back to the action of the large gauge transformations on the functional integral and it is the identification of the "irreducible components" of the functional integral in finite volume that gives rise to the θ-vacua picture.

The interpretation which emphasizes the rôle of gauge invariance as the relevant requirement leading to the θ-states is somewhat misleading in our opinion. In fact, quite generally, *every* functional measure defined on a field algebra $\mathcal{F}$ with (non–trivial) gauge transformations defines a gauge invariant measure when restricted to the observable (gauge invariant) subalgebra $\mathcal{F}_{obs}$, and two functional measures on $\mathcal{F}$ are physically equivalent iff they give rise to the same (gauge invariant) measure on $\mathcal{F}_{obs}$. Since any equivalence class always contains, by general arguments, a measure invariant under gauge transformations on $\mathcal{F}$, the choice of a measure within a fixed class is always equivalent to the choice of a gauge invariant measure, and corresponds in fact to the so called "choice of gauge fixing". This applies in particular to the choice of boundary conditions, which does not affect the locality and gauge invariance properties of the lagrangean density. Moreover, the gauge invariance of a measure do not imply the invariance of the boundary conditions under gauge transformations *defined on boundary variables*, when such boundary transformations do *not* arise from gauge transformations defined on the whole volume. This is the case of transformations which change the winding number of the field configurations, here $A(T) \to A(T) + n_+$, $A(-T) \to A(-T) + n_-$, with $n_+ \neq n_-$.

To formalize the above statements we first need a notion of reducibility of the functional measure in finite volume (where we cannot exploit time translation invariance [16]. Since the euclidean algebra is commutative, reducibility must make reference to the reconstruction of Quantum Mechanics (QM) and in fact it can be formulated by exploiting OS positivity: a functional measure $d\mu_T$ satisfying OS positivity will be said to be QM reducible if it is a convex combination of functional measures which are OS positive. It is not difficult to see that the reducibility of the measure in the above sense is equivalent to the reducibility of the algebra $\mathcal{A}_E$ generated by the time zero algebra and by $\exp{-\tau H}$, $\tau \geq 0$, in the Hilbert space $\mathcal{H}_{OS}$ reconstructed from the finite volume correlation functions through OS positivity.

To discuss the relation between QM reduction and boundary condition, we start with some general preliminary results; we denote by X a (topological) space of configuration at fixed euclidean time, and by $C^0(X)$ the space of continuous functions on X, with the interpretation of the (abelian) field algebra at fixed time.

Theorem 3. [9] *Let* $d\mu_{\xi_-,\xi_+,T}(\xi(\tau))$, $\xi(\tau) \in X$ *be a (complex) functional measure defined by a hermitian kernel* $K_\tau(\xi,\xi')$, *which defines a bounded semigroup in* $L^2(d\xi)$ *(with generator H) irreducible in the sense of Theorem 2, iv); let*

$$d\mu_{\sigma T} \equiv \int d\sigma(\xi_-,\xi_+)\, d\mu_{\xi_-,\xi_+,T}\,(\xi(\tau)) \tag{20}$$

where $d\sigma$ *is a (boundary) measure of the form*

$$d\sigma(\xi_-,\xi_+) = s(\xi_-,\xi_+)d\xi_-\, d\xi_+ \quad , \qquad s(\xi_-,\xi_+) \in L^2(d\xi_-,d\xi_+)$$

Then the correlation functions defined by the functional measure $d\mu_{\sigma T}$ *satisfy the OS positivity iff* $s(\xi_-,\xi_+)$ *is of the form*

$$s(\xi_-,\xi_+) = \sum_i \lambda_i\, \bar{\psi}_i(\xi_-)\, \psi_i(\xi_+) \quad , \quad \psi_i \in L^2, \tag{21}$$

$d\mu_{\sigma T}$ *is then of the form* $\sum \lambda_i\, d\mu_{\psi_i,T}$, *where each* $d\mu_{\psi_i,T}$ *satisfies the OS positivity. The OS reconstructed space* $\mathcal{H}_{\mathcal{OS}}$ *is a direct sum of copies* $\mathcal{H}_i \simeq L^2(X,d\xi)$, *obtained by OS reconstruction with boundary conditions given by* ψ_i; *each* $\mathcal{H}_i$ *is stable under the algebra* $\mathcal{A}_E$ *generated by* $C^0(X)$ *and* $\exp -\tau H$, $\tau > 0$; $\mathcal{A}_E$ *is irreducible in* $\mathcal{H}_i$, *as a consequence of the irreducibility of* K_τ.

Thus, for irreducible boundary conditions, i.e. $s(\xi_-,\xi_+) = \bar{\psi}(\xi_-)\,\psi(\xi_+)$, and only in this case, the integral in the euclidean variables with measure (20) represents the euclidean correlation functions, on the state ψ with wave function $\psi(\xi)$, of a QM model living in $L^2(d\xi)$ with Hamiltonian H:

$$(\psi, e^{-H(T+\tau_1)}\, F_1(\xi)\, e^{-H(\tau_2-\tau_1)} \ldots F_n\, e^{-H(T-\tau_n)}\, \psi) =$$

$$\int \bar{\psi}(\xi_-)\,\psi(\xi_+)\, d\mu_{\xi_-,\xi_+,T}\,(\xi(\tau))\, d\xi_-d\xi_+ \tag{22}$$

The next issue is the decomposition of $\mathcal{H}_{OS}$ into irreducible representations of an "observable subalgebra". In the above general framework, we denote by $C^0_{obs}(X)$ a subalgebra of $C^0(X)$, which will be interpreted as the observable algebra at fixed (euclidean) time.

Theorem 4 [9] *Under the assumptions of Theorem 3, with irreducible* $d\mu_{\sigma T}$, *let* U *be a (non-trivial) operator acting on* $C^0(X)$ *with the properties:*
1) $U : C^0(X) \rightarrow C^0(X)$
2) $[U, K_\tau] = 0$
3) U *commutes with* $C^0_{obs}(X)$, *as multiplication operators.*

Then the restriction of the measure $d\mu_{\sigma T}$ to the algebra generated by $\prod_\tau C^0_{obs}(X)$ is QM reducible, and the algebra $\mathcal{A}_{E,obs}$ generated by $C^0_{obs}(X)$ and $\exp{-\tau H}$, $\tau > 0$ is reducible in $\mathcal{H}_{OS}$. If U is normal (as an operator in $\mathcal{H}_{OS}$), then its spectral projectors reduce the representation of $\mathcal{A}_{E,obs}$, and the corresponding Hilbert space decomposition is obtained by decomposing the boundary wave function $\psi(\xi)$ according to the spectrum of U.

In the $(QED)_{0+1}$ model the rôle of U is played by the unitary operator $\exp iq$ implementing the "large gauge transformations", and the corresponding spectral decomposition of the boundary conditions gives the θ decomposition of $\mathcal{H}_{OS}$. Such decomposition is trivial in the representation of the functional integral in terms of the variables E, q, since in this case U is a multiplication operator in the euclidean variables. In the (φ, A) representation the decomposition is less trivial, and it is given by eq.(10). In fact, the decomposition (10), applied to the wave functions which define the boundary conditions, gives the θ decomposition of the functional integral of variables in the *observable* euclidean algebra, generated by the gauge invariant exponentials $\exp i(2\pi m A(\tau) + n\varphi(\tau))$, $m, n \in \mathbb{Z}$, with an obvious winding number interpretation (for simplicity, here and in the following, we put $\theta_F = 0$):

$$\int_{-\infty}^{\infty} dA_- \, dA_+ \int_0^{2\pi} d\varphi_- \, d\varphi_+ \, \bar{\psi}(\varphi_-, A_-) \, \psi(\varphi_+, A_+)$$

$$\int d\mu_{\varphi_-, A_-, \varphi_+, A_+, T} \, (\varphi(\tau), A(\tau)) \, F_1(\varphi(\tau_1), A(\tau_1)) \ldots F_n(\varphi(\tau_n), A(\tau_n)) =$$

$$= \frac{1}{2\pi} \int_0^1 dB_- \, dB_+ \int_0^{2\pi} d\varphi_- \, d\varphi_+ \int_0^{2\pi} d\theta_- \, d\theta_+ \sum_{n, \nu, k_-, k_+ \in \mathbb{Z}} e^{-i(B_- - \theta_F)(\varphi_- - \theta_- - 2\pi k_-)}$$

$$e^{i(B_+ - \theta_F)(\varphi_+ - \theta_+ - 2\pi k_+)} e^{-in(\theta_+ - \theta_-)} e^{i\nu(\varphi_+ - \theta_+ - 2\pi k_+)} \, F_1 \ldots F_n$$

$$\bar{\phi}(\varphi_- - \theta_- - 2\pi k_-, \theta_-) \, \phi(\varphi_+ - \theta_+ - 2\pi k_+, \theta_+) d\mu_{\varphi_-, B_-, \varphi_+, B_+ + \nu, T} \, (\varphi(\tau), A(\tau) - n) \quad (23)$$

where we have put $A_\pm = B_\pm + n_\pm$, $B_\pm \in [0,1)$, $n_\pm \in \mathbb{Z}$, $n_+ = n_- + \nu \equiv n + \nu$, and used eq.(16). Since $F_1, \ldots, F_n$ are invariant under $A \to A + n$, we can drop n in the argument of the measure, so that the sum on n gives a $\delta(\theta_- - \theta_+)$ and the integration over θ_+, with a relabelling $\theta_- \to \theta$, yields a decomposition of the form

$$\int_0^{2\pi} d\theta \sum_{\nu \in \mathbb{Z}} e^{-i\nu\theta} \, d\mu^{\theta_F}_{\varphi_-, B_-, \varphi_+, B_+, \nu, T} \, (\varphi(\tau), B(\tau))$$

$$\int dB_- \, dB_+ \, d\varphi_- \, d\varphi_+ \, \bar{\psi}_\theta(\varphi_-, B_-)\psi_\theta(\varphi_+, B_+) \ F_1 \dots F_n \tag{24}$$

with $d\mu_{\varphi_-, B_-, \varphi_+, B_+\nu, T}\left(\varphi(\tau), B(\tau)\right)$ the restriction of $d\mu_{\varphi_-, B_-, \varphi_+, B_++\nu, T}\left(\varphi(\tau), A(\tau)\right)$ to the euclidean observable algebra, with $B(\tau) = A(\tau) \mod 1$.

A sharp isolation of a θ state representation requires to choose, as boundary condition, a wave function $\psi_\theta(\varphi, A)$ in the space of the representation π_0 described in Theorem 1,

$$\psi_\theta(\varphi, A) = \sum_n e^{i(A-\theta_F)(\varphi-\theta-2\pi n)} \, \Phi(\varphi - \theta - 2\pi n) \tag{25}$$

with $\Phi \in L^2$; one must then integrate with respect to $d\varphi_- \, d\varphi_+ \, dA_+$ and take the *ergodic mean* in A_-. This shows that the construction of θ states (in finite volume) requires boundary conditions which are crucially "non–local" in the variable A. The construction is very close to that of Bloch θ sectors for a particle in a periodic potential [8].

It is worthwhile to stress that the above decomposition into pure phases of $\mathcal{A}_{E,obs}$ is done in finite volume and it is radically different from the usual phase decompositions, which are obtained in Statistical Mechanics and (euclidean) Field Theory *after* the infinite volume limit [16]. This makes the discussion of symmetry breaking very different from the standard case, since the rôle of the thermodynamical limit is substantially different.

The above decomposition of the functional integral in finite volume has a winding number interpretation. The integer ν is the winding number which classifies the trajectories of A as trajectories on the spectrum of $\exp 2\pi i A$, which is a circle, exactly as in the case of a particle in a periodic potential $W(x)$, with the periodic functions of x playing the rôle of the observables [8]. In QED_{0+1} such topological structure crucially depends on the compactness of the chiral group, since otherwise no non–trivial functions of A would be observable. As in the case of periodic potentials, it is easy to see that the topological numbers of the configurations do not remain bounded in the infinite volume limit, and therefore the discussion in finite volume is essential for the use of the topological classification of the configurations.

We may now easily discuss the effect of the addition of a (gauge invariant) "topological term" to the Lagrangean, namely a term $\theta_L \dot{A}$, leading to an interaction of the form

$$i\theta_L \int_{-T}^T \dot{A}(\tau)\, d\tau = i\theta_L(A(T) - A(-T)) \tag{26}$$

in the euclidean action. From the above discussion it follows that the effect on the measure $d\mu(\varphi(\tau), A(\tau))$ is a change of boundary conditions

$$d\sigma(\varphi_-, A_-, \varphi_+, A_+) \to e^{i\theta_L (A_+ - A_-)} d\sigma(\varphi_-, A_-, \varphi_+, A_+) \qquad (27)$$

This means that the addition of the topological term amounts to a change of boundary conditions $\psi(\varphi, A) \to \exp(i\theta_L A)\,\psi(\varphi, A)$; in particular, for boundary conditions (25) leading to a θ state, the effect of the term (26) is to yield the sector labelled by $\theta + \theta_L$.

Thus, the standars link between the lagrangean parameter θ_L and the parameter θ, which labels the irreducible representations of the observable algebra, only holds if the boundary conditions are those of eq.(25), with $\theta = 0$. For *generic boundary conditions*, which define reducible representations of the observable algebra, the addition of the topological term only shifts the support of the θ reduction; as we will see, for $M \neq 0$, the infinite volume limit *removes* such a reducibility and selects (generically in the boundary conditions) the sector with $\theta = \theta_M$, *independently of the topological term*.

The above considerations do not apply to the formulation based on a functional integral on the spectrum of the euclidean *observable* algebra; there, the addition of the topological term (l.h.s. of eq.(26)) does not reduce to a change of boundary conditions because $\exp i\theta_L A(\pm T)$ is not an observable, and eq.(26) is not available. Therefore, in that strategy, the construction of θ sectors is not related to the decomposition of the boundary conditions; in fact the angle θ enters directly in the measure (through the the kernel of $\exp -\tau H$) as a free parameter.

4. Thermodynamical limit. Convergence to the ground state with $\theta = \theta_M$ in the massive case.

As usual, the construction of the lowest energy state(s) requires the control of the thermodynamical limit $(T \to \infty)$ of properly normalized correlation functions, eq.(22). Unlike the standard case (of models with strictly positive kernels, with positive boundary conditions), here the so constructed state depends in general on the boundary conditions (b.c.). We concentrate our discussion on the limit of the correlation functions of observables. The main problem is whether the reduction into sectors, which appears in finite volume for generic b.c., survives the $T \to \infty$ limit. For the case $M \neq 0$, the situation is very similar to that of particle in a periodic potential, since the lowest energy state is for small M unique (Theorem 1); this implies (even if the ground state does not belong to the space of the OS reconstruction in finite volume, see Theorem 1) that for generic b.c. the correlation functions converge to the

18

expectations on the unique ground state, which belongs to the sector $\mathcal{H}_{\theta=\theta_M}$ (*no CP violation*).

Theorem 5. $\forall f(\varphi, A) \in (L^1 \cap L^2)([0, 2\pi) \times \mathbb{R}, d\varphi \, dA)$, *such that*

$$f_\theta^0 \equiv (\psi_\theta^0, f)_{L^2([0,2\pi) \times \mathbb{R}, d\varphi \, dA)} \tag{28}$$

with $\psi_\theta^0(\varphi, A)$ *the wave function of the lowest energy state in* $\mathcal{H}_\theta$, *does not vanish for* $\theta = \theta_M$, *and for all observables* $F_1 \ldots F_n$, $F_k = \exp i(2\pi m_k A(\tau_k) + l_k \varphi(\tau_k))$ *one has*

$$\lim_{T \to \infty} Z(f,T)^{-1} \int_{-\infty}^{\infty} dA_- \, dA_+ \int_0^{2\pi} d\varphi_- \, d\varphi_+ \bar{f}(\varphi_-, A_-) \, f(\varphi_+, A_+)$$

$$\int d\mu_{\varphi_-, A_-, \varphi_+, A_+, T} \, (\varphi(\tau), A(\tau)) \, F_1 \ldots F_n =$$

$$= (\psi_{\theta_M}^0, e^{i(2\pi m_1 A + l_1 \varphi)} e^{-(\tau_2 - \tau_1)(H - E^0(\theta_M))} \ldots e^{i(2\pi m_n A + l_n \varphi)} \psi_{\theta_M}^0) \tag{29}$$

for M *small. For* $M = 0$, *the above limit gives*

$$\int_0^{2\pi} d\theta \, g(\theta) \, (\psi_\theta^0, e^{i(2\pi m_1 A + l_1 \varphi)} e^{-(\tau_2 - \tau_1)(H - E^0(\theta))} \ldots e^{i(2\pi m_n A + l_n \varphi)} \psi_\theta^0) \tag{30}$$

with $g(\theta) \equiv |f_\theta^0|^2 / \int_0^{2\pi} |f_q^0|^2 \, dq$. *The normalization constant* $Z(f, T)$ *is given by the functional integral for* $F_k = 1$, $k = 1, \ldots, n$.

Proof. The proof is similar to that for a particle in a periodic potential [8] (for more details see [9]). The essential ingredients are: i) the Feynman–Kac representation, eq.(22), ii) the finite volume decomposition of the (OS) Hilbert space into θ sectors, iii) the discreteness of the spectrum of H in each sector $\mathcal{H}_\theta$, iv) the continuity in θ of $|f_\theta^0|$, which gives the weight of the component of $f_\theta(\varphi, A)$, eq.(10), over the ground state $\psi_\theta^0(\varphi, A)$, v) the uniqueness of the ground state $\psi_{\theta_M}^0$ (Theorem 1), and the non–zero gap $\inf_{\theta, n > 0}(E^n(\theta) - E^0(\theta)) \equiv \delta > 0$, for $M \neq 0$ and small. In fact, the l.h.s. of eq. (29) is of the form

$$[\int_0^{2\pi} d\theta \, |f_\theta^0|^2 \, (\psi_\theta^0, e^{-(T+\tau_1)H} e^{i(2\pi m_1 A + l_1 \varphi)} \ldots e^{-(T-\tau_n)H} \psi_\theta^0) + O(e^{-2(E^0(\theta) + \delta)T})]$$

$$[\int_0^{2\pi} d\theta \, |f_\theta^0|^2 \, (\psi_\theta^0, e^{-2TH} \psi_\theta^0) + O(e^{-2(E^0(\theta) + \delta)T})]^{-1}. \tag{31}$$

For large T, since $|f_\theta^0|$ is continuous and $|f_{\theta_M}^0| \neq 0$ only the first terms in the square brakets survive. Moreover,

$$(\psi_\theta^0, e^{-(T-\tau_1)H} e^{i(2\pi m_1 A + l_1 \varphi)} \dots e^{i(2\pi m_n A + l_n \varphi)} e^{-(T-\tau_n)H} \psi_\theta^0) = e^{-2E^0(\theta)T} G(\theta) \quad (32)$$

with $G(\theta)$ the correlation function with a renormalized Hamiltonian $H \to H - E^0(\theta)$; since $E^0(\theta) > E^0(\theta_M)$, $\forall \theta \neq \theta_M$,

$$|f_\theta^0|^2 \, e^{-2E^0(\theta)T} [\int_0^{2\pi} d\theta \, |f_\theta^0|^2 e^{-2E^0(\theta)T}]^{-1} \to \delta(\theta - \theta_M) \quad (33)$$

for $T \to \infty$ in the sense of measures, and eq.(29) follows since $G(\theta)$ is continuous in θ.

For $M = 0$, the result follows immediately from the fact that $E^0(\theta)$ is independent of θ, so that the r.h.s. of eq.(32) becomes $\exp{-2E^0 T} \, G(\theta)$, and the l.h.s. of eq.(33) is independent of T, and actually given by $g(\theta)$. (The normalization factor $Z(f, T)$ is strictly positive, since it is given by the l.h.s. of eq.(22), with $F_i = 1$, and $\exp(-\tau H) > 0$).

The above proof makes clear the substantial difference between the $M = 0$ and the $M \neq 0$ cases and shows that the limit $M \to 0$ does not commute with the thermodynamical limit ($T \to \infty$) [9][13]. Since the essential ingredient is the existence (for $M \neq 0$) of a unique absolute minimum of $E^0(\theta)$, the above result applies also to higher dimensions on the basis of a θ sector decompositions of the form of eqs.(24),(31) with the free energy density in the θ sector playing the rôle of $E^0(\theta)$.

In contrast with the approach which restricts the attention to the algebra of observables, the strategy discussed above, which integrates over lagrangean field variables, allows to compare the free energy density for different values of θ, and to exploit the fact that it has a (unique) minimum, at $\theta = \theta_M$.

The implications on the strong CP problem is that in the first case the renormalization is done *after* having fixed the value of θ (which therefore appears as a free parameter), whereas in the second case the renormalization automatically preserves the property that the free energy density has a unique minimum, at $\theta = \theta_M$, which holds generically in the values of the parameters, so that a "reduction" automatically take place in the infinite volume limit, with the selection of $\theta = \theta_M$ (*natural strong CP symmetry*).

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